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Infinitesimal symmetry transformations of the Langevin equation

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Abstract. The Lie algebra associated with infinitesimal symmetry transformations of the equation of motion for a radiating charged particle embedded in the radiation field, i.e. the so-called Langevin equation, is determined. It is shown that the invariance group is a five-parameter group with a commutative three-parameter subgroup.

1. Introduction

If we neglect the force due to the magnetic radiation as is usual in many non-relativistic calculations, then the one-dimensional motion equation of a particle in the presence of radiation is (de la Peña-Auerbach and Cetto 1977)

$$m\ddot{x} = eE + F + m\tau\dddot{x}, \quad \tau = 2e^2/(3mc^3) \tag{1}$$

where m is the mass, e the charge of the particle, $F = F(x, t)$ the external given force and E the electric field of the radiation.

Equation (1) is a stochastic differential equation which can be solved, in principle, provided we known the properties of the stochastic process $E(t)$. It is a fundamental equation (the so-called Langevin equation) of the stochastic electrodynamics (de la Peña-Auerbach and Cetto 1977, Santos 1974). Hence it is of interest to study some features of dynamical systems described by this equation. In the present paper we examine the infinitesimal invariance transformations. Several methods (Gonzales-Gascon 1977, Leach 1981, Lutzky 1978, Anderson and Davison 1974, Aguirre and Krause 1984) can be found in the current literature to study the invariance properties of differential equations. We shall use an extension of the method used by Aguirre and Krause (1984).

2. Infinitesimal symmetry transformations

We are interested in the invariance properties of the model Langevin equation. From equation (1) we can write with the usual notation for time derivatives

$$\dddot{x} = (1/\tau)\ddot{x} - (1/m\tau)F - (e/m\tau)E(t) \tag{2}$$

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For a force F linear in x ($F=-Kx$, $K=\text{constant}$, $K>0$), we obtain

$$\dddot{x}=(1/\tau)\ddot{x}+(K/m\tau)x-(e/m\tau)E(t) \tag{3}$$

where τ, by equation (1), is finite.

We consider the symmetries generated by infinitesimal transformations of the form

$$t'=t+\varepsilon N(x,t) \tag{4}$$

$$x'=x+\varepsilon T(x,t) \tag{5}$$

where ε denotes a parameter such that $0<\varepsilon\ll 1$.

From the assumed invariance of equation (3) under the infinitesimal transformations (4) and (5), we obtain that $T(x,t)$ and $N(x,t)$ are such that

$$T(x,t)=\Omega_1(t) \tag{6}$$

$$N(x,t)=\Omega_2(t)x+\Omega_3(t) \tag{7}$$

where $\Omega_i(t)$ $(i=1,2,3)$ satisfy the equations

$$\dddot{\Omega}_1(t)-(1/3\tau^2)\dot{\Omega}_1(t)=0 \tag{8}$$

$$3\dot{\Omega}_2(t)=3\ddot{\Omega}_1(t)+(1/\tau)\dot{\Omega}_1(t) \tag{9}$$

$$\dddot{\Omega}_3(t)-(1/\tau)\ddot{\Omega}_3(t)-(K/m\tau)\Omega_3(t) = -(3e/m\tau)E(t)\dot{\Omega}_1(t)-(e/m\tau)\dot{E}(t)\Omega_1(t)-(e/m\tau)E(t)\Omega_2(t) \tag{10}$$

and the binding relation

$$\dddot{\Omega}_2(t)-(1/\tau)\ddot{\Omega}_2(t)-(3K/m\tau)\dot{\Omega}_1(t)=0. \tag{11}$$

The system of equations (8)–(10) has seven linearly independent solutions. Relation (11) is an additional condition; it restricts in some cases the number of linearly independent solutions.

3. The Lie algebra associated with infinitesimal symmetry transformations

It is known that the infinitesimal generators X_a ($a=1,2,\ldots,N$ where N is the number of parameters) of a Lie group satisfy the relation

$$[X_a,X_b]=f^c_{ab}X_c \tag{12}$$

where f^c_{ab} are the structure constants, the square brackets denote antisymmetrisation of the indices a and b, and the usual convention of summation on repeated indices is used.

It is also known that in the (t,x) realisation and in some α parametrisation the infinitesimal generator can be written as

$$X_a(x,t)=N_a(x,t)\partial/\partial t+T_a(x,t)\partial/\partial x \tag{13}$$

where in the present case $a=1,2,\ldots,7$, because the maximum number of parameters in the Lie group which leaves an ordinary differential equation of the third-order invariant is seven (Cohen 1931).

Substituting (13) into (12) and separating the coefficients of $\partial/\partial t$ and $\partial/\partial x$, we find that ($G_{k;r}=\partial G_k/\partial r$; $G=N, T$; $k=a, b$; $r=x, t$)

$$f^c_{ab}N_c = N_aN_{b;t} + T_aN_{b;x} - N_bN_{a;t} - T_bN_{a;x} \tag{14}$$

and

$$f^c_{ab}T_c = N_aT_{b;t} + T_aT_{b;x} - N_bT_{a;t} - T_bT_{a;x}. \tag{15}$$

By means of expressions (6) and (7) for N and T, we obtain from (14) and (15) the following equations to determine all structure constants:

$$f^c_{ab}\Omega_{1.c} = \Omega_{1.a}\dot{\Omega}_{1.b} - \Omega_{1.b}\dot{\Omega}_{1.a} \tag{16}$$

$$f^c_{ab}\Omega_{2.c} = \Omega_{1.a}\dot{\Omega}_{2.b} - \Omega_{1.b}\dot{\Omega}_{2.a} \tag{17}$$

$$f^c_{ab}\Omega_{3.c} = \Omega_{1.a}\dot{\Omega}_{3.b} + \Omega_{3.a}\Omega_{2.b} - \Omega_{1.b}\dot{\Omega}_{3.a} - \Omega_{3.b}\Omega_{2.a} \tag{18}$$

$$f^c_{ab}\dot{\Omega}_{1.c} = \dot{\Omega}_{1.a}\dot{\Omega}_{1.b} + \Omega_{1.a}\ddot{\Omega}_{1.b} - \dot{\Omega}_{1.b}\dot{\Omega}_{1.a} - \Omega_{1.b}\ddot{\Omega}_{1.a} \tag{19}$$

$$f^c_{ab}\ddot{\Omega}_{1.c} = \dot{\Omega}_{1.a}\ddot{\Omega}_{1.b} + \Omega_{1.a}\dddot{\Omega}_{1.b} - \dot{\Omega}_{1.b}\ddot{\Omega}_{1.a} - \Omega_{1.b}\dddot{\Omega}_{1.a} \tag{20}$$

$$\begin{aligned} f^c_{ab}\dot{\Omega}_{3.c} = {} & \dot{\Omega}_{1.a}\dot{\Omega}_{3.b} + \Omega_{1.a}\ddot{\Omega}_{3.b} + \dot{\Omega}_{3.a}\Omega_{2.b} + \Omega_{3.a}\dot{\Omega}_{2.b} - \dot{\Omega}_{1.b}\dot{\Omega}_{3.a} \\ & - \Omega_{1.b}\ddot{\Omega}_{3.a} - \dot{\Omega}_{3.b}\Omega_{2.a} - \Omega_{3.b}\dot{\Omega}_{2.a} \end{aligned} \tag{21}$$

$$\begin{aligned} f^c_{ab}\ddot{\Omega}_{3.c} = {} & \ddot{\Omega}_{1.a}\dot{\Omega}_{3.b} + 2\dot{\Omega}_{1.a}\ddot{\Omega}_{3.b} + \Omega_{1.a}\dddot{\Omega}_{3.b} + \ddot{\Omega}_{3.a}\Omega_{2.b} + 2\dot{\Omega}_{3.a}\dot{\Omega}_{2.b} \\ & + \Omega_{3.a}\ddot{\Omega}_{2.b} - \ddot{\Omega}_{1.b}\dot{\Omega}_{3.a} - 2\dot{\Omega}_{1.a}\ddot{\Omega}_{3.a} \\ & - \Omega_{1.b}\dddot{\Omega}_{3.a} - \ddot{\Omega}_{3.b}\Omega_{2.a} - 2\dot{\Omega}_{3.b}\dot{\Omega}_{2.a} - \Omega_{3.b}\Omega_{2.a} \end{aligned} \tag{22}$$

where $\Omega_{i.a}$ ($i=1, 2, 3$; $a=1, 2, \ldots, 7$) denotes function $\Omega_i(t)$ corresponding to the parameter a.

Equations (16)-(22) hold for all t. Thus we consider them at $t=0$ and this leads to the initial value problem of $N_a(x, t)$ and $T_a(x, t)$ ($a=1, 2, \ldots, 7$). In order to analyse this problem, we note the general expressions for $N(x, t)$ and $T(x, t)$ in some α-parametrisation are

$$N(x, t) = \alpha^a\Omega_{1.a}(t) \tag{23}$$

$$T(x, t) = \alpha^a(\Omega_{2.a}(t)x + \Omega_{3.a}(t)). \tag{24}$$

Since α^a are arbitrary parameters in (23) and (24) we can choose the values ($G_r=\partial G/\partial r$, $G_{rr}=\partial^2 G/\partial r^2$; $G=N, T$; $r=x, t$)

$$\alpha^1 = N(0, 0) \qquad \alpha^2 = N_t(0, 0) \qquad \alpha^3 = N_{tt}(0, 0) \qquad \alpha^4 = T(0, 0)$$

$$\alpha^5 = T_t(0, 0) \qquad \alpha^6 = \tfrac{1}{2}T_{tt}(0, 0) \qquad \alpha^7 = T_x(0, 0)$$

and use these α as essential parameters of the group. It follows from (23) and (24) that

$$\Omega_{1.a}(0) = \delta_{a.1} \qquad \dot{\Omega}_{1.a}(0) = \delta_{a.2} \qquad \ddot{\Omega}_{1.a}(0) = \delta_{a.3}$$

$$\Omega_{2.a}(0) = \delta_{a.7} \qquad \Omega_{3.a}(0) = \delta_{a.4} \qquad \dot{\Omega}_{3.a}(0) = \delta_{a.5}$$

$$\ddot{\Omega}_{3.a}(0) = 2\delta_{a.6}. \tag{25}$$

With results (25) and $\dddot{\Omega}_{1.a}(0)$, $\dot{\Omega}_{2.a}(0)$, $\ddot{\Omega}_{2.a}(0)$, $\dddot{\Omega}_{3.a}(0)$ determined by (8)-(10) for $t=0$, the structure constants (and hence the Lie algebra) are obtained from (16)-(22). We have the following non-null structure constants

$$f^4_{15} = 1 \qquad f^4_{47} = 1 \qquad f^5_{16} = 2 \qquad f^5_{57} = 1 \qquad f^6_{14} = K/2m\tau$$

$$f^6_{16} = 1/\tau \qquad f^6_{17} = eE(0)/2m\tau \qquad f^6_{67} = 1.$$

Table 1. Lie algebra for equation $\dddot{x}+(e/m\tau)E(t)-(k/m\tau)x=(1/\tau)\ddot{x}$. See text for explanation.

	X_1	X_4	X_5	X_6	X_7
X_1	0	$\frac{1}{2}(k/m\tau)X_6$	X_4	$2X_5+(1/\tau)X_6$	$(e/2m\tau)E(0)X_6$
X_4	$-\frac{1}{2}(k/m\tau)X_6$	0	0	0	X_4
X_5	$-X_4$	0	0	0	X_5
X_6	$-2X_5-(1/\tau)X_6$	0	0	0	X_6
X_7	$-(e/2m\tau)E(0)X_6$	$-X_4$	$-X_5$	$-X_6$	0

Using the structure constants enumerated above and (12) and (13) we obtain the Lie algebra presented in table 1 where one gets the commutator $[X_a, X_b]$ at the intersection of the ath row and the bth column.

From table 1, it follows that the Lie algebra associated with the symmetry group of the one-dimensional Langevin equation is generated by five infinitesimal operators, and that it contains one commutative subalgebra, i.e. (X_4, X_5, X_6). Hence, we have the result that the symmetry group which preserves the invariance of equation (3) is a five-parameter Lie group with a commutative three-parameter subgroup.

4. Concluding remarks

The Langevin equation belongs to a class of differential equations $\dddot{x}=g(x,\dot{x},\ddot{x},t)$ with a convenient choice of coefficients. In principle, it is possible with the procedure used here to determine the symmetry group for all equations of this class. However, equation (3) has a direct and known physical interest. For this reason we have studied equation (3) specifically. We have determined the Lie algebra associated with infinitesimal point symmetries and found that the invariance group is a five-parameter Lie group with an Abelian three-parameter subgroup. It is interesting to note that by the present approach to obtain Lie algebras one uses the value of the function $E(t)$ at $t=0$ and it is not necessary to known the explicit expressions of the infinitesimal generators. The generators X_a are obtained by considering the solutions $\Omega_i(t)$ $(i=1,2,3)$, $t\neq 0$, of equations (8)–(11). Since equation (10) depends on the radiation field $E(t)$ and on $\dot{E}(t)$, the process of determining $\Omega_i(t)$ can represent a difficult problem. A study of this subject will be reported in a future paper.

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